Prof. Vidya Sagar
Department of physics
B.Sc (Part-I) Hons

Paper-I

Poiseuille's Formula for liquid flow through a Capillary tube.

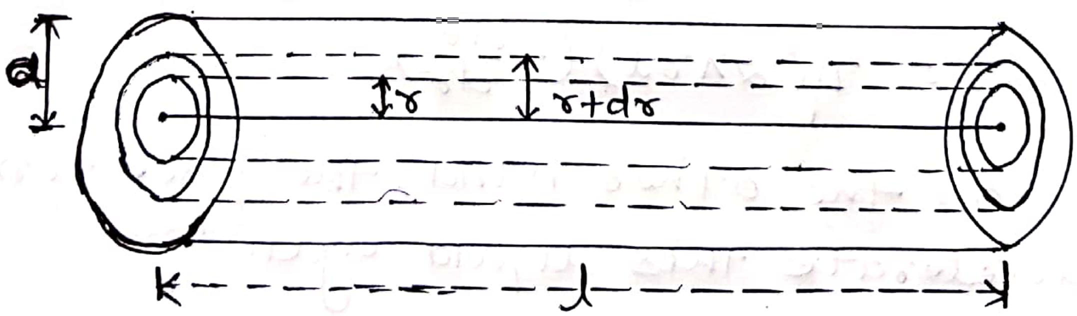
Poiseuille's derived an expression for the volume of a liquid flowing per Second through a horizontal tube of uniform bore under a constant pressure difference between its end. He made the fallowing assumptions:-

(i) The flow is streamlined and parallel to the axis of the tube.

(ii) The flow is steady, i.e. there is no acceleration of the liquid at any point.

(iii) There is no radial flow, i.e. the pressure over any cross-section is constant.

(iv) The liquid in contanct with the wall of the tube is at rest.



Let us consider a horizontal tube of length l and radius a. Let a liquid be maintained flowing through it under a constant pressure difference p applied its ends. The velocity of the liquid flowing in the tube is a maximum along the axis and is zero at the wall of the tube. Suppose, in the steady state, the velocity of the cylindrical liquid layer at a distance r from the axis of the tube is v. Evidently, at every point of the liquid layer, the velocity gradient perpendicular to the direction of flow is $-\frac{dv}{dr}$.

Let us now consider the force acting on the liquid cylinder of radius r. The liquid outside this cylinder is flowing with a smaller velocity and therefore, exerts a backward viscous force on the cylinder. By Newton's hypothesis, the viscous force,

= coeff. of viscosity × surface area of the layer × velocity gradient

$$= \eta(2\pi r l)\left(-\frac{dv}{dr}\right)$$

On the other hand, the force tending to accelerate this liquid cylinder

= pressure difference between its end × area of cross-section

$$= p(\pi r^2)$$

Since there is no acceleration of the liquid these force must balance, so that

$$P\pi r^2 = -\eta(2\pi r l)\frac{dv}{dr}$$

or, $$\frac{dv}{dr} = -\frac{P}{2\eta l}r$$

or, $$dv = -\frac{P}{2\eta l}r\,dr$$

Integrating, we obtain

$$\int dv = -\frac{P}{2\eta l}\int r\,dr$$

$$v = -\frac{P}{2\eta l}\left(\frac{r^2}{2}\right) + A \quad \text{—— (1)}$$

Where A is a constant of integration whose value is determined from the boundary condition. At the walls of the tube the velocity is zero i.e. at $x = a$, $v = 0$. putting these values in the equation (1), we get

$$0 = -\frac{P}{2\eta l}\left(\frac{a^2}{2}\right) + A$$

$$\therefore A = \frac{P}{2\eta l}\left(\frac{a^2}{2}\right)$$

$$\therefore v = -\frac{P}{2\eta l}\left(\frac{r^2}{2}\right) + \frac{P}{2\eta l}\left(\frac{a^2}{2}\right)$$

$$\therefore v = \frac{P}{2\eta l}\left(\frac{a^2}{2} - \frac{r^2}{2}\right)$$

$$v = \frac{P}{4\eta l}(a^2 - r^2) \quad \text{—— (2)}$$

This is the equation of a parabola. It represents the velocity distribution in the streamline flow of the liquid with respect to the axis of the tube. It is dependent of the length l of the tube, because P/l is the pressure drop per unit length and has the same value at any point along the length of the tube.

Let us now consider a thin cylendorical shell of radii r and $r+dr$. The volume of the liquid flowing per second through it is given by

dQ = velocity x cross-sectional area of the shell.

$$= v(2\pi r\,dr)$$

$$= \frac{P}{4\eta l}(a^2-r^2)(2\pi r\,dr)$$

The volume of the liquid, Q, flowing per second through the whole tube is obtained by integrating this expression between the limit $r=0$ and $r=a$

$$\therefore Q = \int_0^a \frac{P}{4\eta l}(a^2-r^2)\,2\pi r\,dr$$

$$= P\cdot\frac{2\pi}{4\eta l}\int_0^a (a^2 r - r^3)\,dr$$

$$= \frac{\pi P}{2\eta l}\left[a^2\frac{r^2}{2} - \frac{r^4}{4}\right]_0^a$$

$$= \frac{\pi P}{2\eta l}\left[\frac{a^4}{2} - \frac{a^4}{4}\right]$$

$$= \frac{\pi P}{2\eta l}\left[\frac{2a^4 - a^4}{4}\right]$$

$$\therefore \boxed{Q = \frac{\pi P a^4}{8\eta l}}$$

This is known as poiseuille's formula